

§1. Compactness

Covers

A class of subsets $H = \{\Omega_j\}$ of E is called a cover of a set G of E if we have

$$G \subset \bigcup_j \Omega_j$$

Example 1

Let H be a class of subsets given by $H = \{\Omega_j = [j, j + 1[, j \in \mathbb{Z}\}$. Then H is a cover of the real line \mathbb{R} because for every point in \mathbb{R} there exists an interval Ω_j of H contains this point.

Example 2

Let H be a class of subsets given by $H = \{\Omega_j =]j, j + 1[, j \in \mathbb{Z}\}$. Then H is not a cover of the real line \mathbb{R} because the points j in \mathbb{R} do not belong to any interval Ω_j of H .

Open covers

The cover $H = \{\Omega_j\}$ of G is called an open cover if each Ω_j is open.

Example 3

Let H be a class of open subsets given by $H = \{\Omega_j =]j, j + 2[, j \in \mathbb{Z}\}$. Then H is an open cover of the real line \mathbb{R} because for every point in \mathbb{R} there exists an open interval Ω_j of H contains this point.

Finite covers

The cover $H = \{\Omega_{j_1}, \Omega_{j_2}, \dots, \Omega_{j_n}\}$ of G is called a finite cover of G if

$$G \subset \Omega_{j_1} \cup \Omega_{j_2} \cup \dots \cup \Omega_{j_n}.$$

Theorem 1 (Heine-Borel)

Every open cover $H = \{\Omega_j\}$ of a closed and bounded interval $G = [a, b]$ has a finite cover. That is to say, $G \subset \bigcup_{j \in J} \Omega_j$ involves that there exists a finite number of an open subsets $\Omega_{j_1}, \Omega_{j_2}, \dots, \Omega_{j_n}$ such that

$$G \subset \Omega_{j_1} \cup \Omega_{j_2} \cup \dots \cup \Omega_{j_n}.$$

Compact set

A subset G of a topological space E is called compact if every open cover $H = \{\Omega_j\}$ of G contains a finite subcover of G . In other words, for every family $\{\Omega_j\}$, $j \in J$, of an open sets with the property

$$G \subset \bigcup_{j \in I} \Omega_j,$$

there exists a finite subfamily $\{\Omega_{j(k)}\}$, $j(k) \in J$, $k = 1, 2, \dots, n$, such that

$$G \subset \bigcup_{k=1}^n \Omega_{j(k)},$$

or still, there exists a finite number of open subsets $\Omega_{j(1)}, \Omega_{j(2)}, \dots, \Omega_{j(n)} \in H$ such that

$$G \subset \Omega_{j(1)} \cup \Omega_{j(2)} \cup \dots \cup \Omega_{j(n)}.$$

Remark 1

The idea of the compactness is motivated by the Heine-Borel theorem of a closed and bounded interval.

Theorem 2

Every closed and bounded real interval $G = [a, b]$ is compact.

Remark 2

The open bounded interval $]a, b[$, the open closed bounded interval $]a, b]$ and the closed open bounded interval $[a, b[$ are not compact.

Theorem 3

Every finite subset $G = \{\varphi_1, \varphi_2, \dots, \varphi_n\}$ of a topological space E is compact

Theorem 4

A closed subset V of a compact set G is also compact

Proof

Let $H = \{\Omega_j\}$ be an open cover of the closed subset $V \subset G$. That is to say, $V \subset \bigcup \Omega_j$, from the relation $G = V \cup V^c$ and the compactness of G , we get an open cover of the set G , say $G \subset \bigcup \Omega_j \cup V^c$. Hence there exists a finite cover of G

$$G \subset \Omega_{j_1} \cup \Omega_{j_2} \cup \dots \cup \Omega_{j_n} \cup V^c,$$

or still

$$V \cup V^c \subset \Omega_{j_1} \cup \Omega_{j_2} \cup \dots \cup \Omega_{j_n} \cup V^c.$$

Due to the disjointedness of the sets V and V^c , the subfamily $\{\Omega_{j_1}, \Omega_{j_2}, \dots, \Omega_{j_n}\}$ represents a finite cover of the subset V . In other words

$$V \subset \Omega_{j_1} \cup \Omega_{j_2} \cup \dots \cup \Omega_{j_n}. \text{ Hence } V \text{ is compact.}$$

Remark 4

The subset V must be closed, in order to obtain an open subset V^c as an element of the covers of G .

Sequentially compact set

A subset G of a topological space E is called sequentially compact if and only if, every sequence of elements of G contains a subsequence which converges to an element in G .

Remark 5

There is no relation between the compactness and the sequential compactness in the topological space.

Indeed, because the adherent point of a set G in the topological space is not necessary a limit of convergent subsequence.

Remark 6

An adherent point of a set G in a normed space E is a limit of convergent subsequence.

Theorem 5

Let G be a subset of a normed space E . Then the following statements are equivalent

1. G is compact
2. G is sequentially compact

Remark 7

Every bounded sequence (x_n) in the closed bounded interval $G = [a, b]$ has a convergent subsequence (x_{n_k}) in G .

Totally bounded set

A subset G of a normed space E is called a totally bounded if for each real positive $\varepsilon > 0$, there exists a finite number of elements $\varphi_1, \varphi_2, \dots, \varphi_n$ in G such that

$$G \subset \bigcup_{j=1}^n B(\varphi_j, \varepsilon).$$

Net for set

A finite set of points $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ of a normed space E is called ε -net for a subset G of E , if for every point ψ in G there exists a φ_j such that

$$\|\psi - \varphi_j\| < \varepsilon$$

Remark 7

Each element ψ from a totally bounded set G has a distance less than ε from least one of the elements $\varphi_1, \varphi_2, \dots, \varphi_n$. In other words, the totally bounded set G has a ε -net for all $\varepsilon > 0$.

Lemma 1

Each sequentially compact set G is totally bounded

Proof

Indeed, suppose that G is not totally bounded then there exist a real positive $\varepsilon > 0$ and a point $\psi_1 \in G$ with the property

$$\|\psi_1 - \varphi_j\| \geq \varepsilon \quad \text{for all } j = 1, 2, \dots, n.$$

Then there exists $\psi_2 \in G$ such that

$$\|\psi_1 - \psi_2\| \geq \varepsilon \quad \text{and} \quad \|\psi_2 - \varphi_j\| \geq \varepsilon \quad \text{for all } j = 1, 2, \dots, n,$$

for otherwise $\{\varphi_1, \varphi_2, \dots, \varphi_n, \psi_1\}$ would be ε -net for G . Continuing in this manner, we arrive at a sequence $\{\psi_1, \psi_2, \dots, \psi_n, \dots\}$ with the property that

$$\|\psi_i - \varphi_j\| \geq \varepsilon \quad \text{for all } i \neq j.$$

Thus, the sequence (ψ_n) does not contain a convergent subsequence. In other words G is not sequentially compact.

Remark 8

The compact sets G of a normed space E are bounded, closed and complete.

Relatively compact set

A subset G of a normed space E is called relatively compact if its closure \overline{G} is compact.

Remark 9

A subset G of a normed space E is relatively compact if and only if each sequence φ_n of elements from G contains a convergent subsequence. Further, relatively compact sets are totally bounded.

Theorem 6

A bounded and finite dimensional subset G of a normed space E is relatively compact

Proof

This follows from the Bolzano-Weierstrass theorem.

Compactness in $C(K)$

The set $C(K)$ designates the space of continuous real or complex valued functions defined in a compact set $K \subset \mathbb{R}^n$, furnished with the maximum norm

$$\|\varphi\|_\infty = \max_{x \in K} |\varphi(x)|.$$

Theorem 7 (Arzela-Ascoli)

A set $G \subset C(K)$ is relatively compact if and only if it is bounded and equicontinuous. Say, if there exists a constant M such that

$$|\varphi(x)| \leq M \quad \text{for all } x \in K \quad \text{and all } \varphi \in G.$$

Further, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $\varphi \in G$, we have

$$|\varphi(x) - \varphi(y)| < \varepsilon \quad \text{for all } x, y \in K \quad \text{with } |x - y| < \delta.$$

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