§5. Regularization

Left regularizator

Let T be a bounded linear operator defined from a normed space E into a normed space F. Saying a left regularizator of T denoted by R_l all bounded linear operator defined from F into E such that

$$R_l\left(T\varphi\right) = \left(I - A_l\right)\varphi, \tag{1}$$

where A_l is a compact linear operator defined from E into itself.

$$E \xrightarrow{T} F \xrightarrow{R_l} E$$
$$\varphi \mapsto T\varphi \mapsto R_l\left(T\varphi\right)$$

Right regularizator

Let T be a bounded linear operator defined from a normed space E into a normed space F. Saying a right regularizator of T denoted by R_r all bounded linear operator defined from F into E such that

$$T\left(R_r\psi\right) = \left(I - A_r\right)\psi, \tag{2}$$

where A_r is a compact linear operator defined from F into itself.

$$F \xrightarrow{R_r} E \xrightarrow{T} F$$
$$\psi \mapsto R_r\psi \mapsto T\left(R_r\psi\right)$$

Regularizators

Let T be a bounded linear operator defined from a normed space E into a normed space F. Saying a regularizator of T denoted by R all bounded linear operator defined from F into E such that

$$RT = I - A_l,$$

and

$$TR = I - A_r,$$

where A_l and A_r are two linear operators defined from E into E and from F into F respectively.

Theorem 1

The difference between the left regularizator R_l and the right regularizator R_r is a compact operator.

Proof

According to the relations (1) and (2), we have

$$R_lT = I - A_l.$$

The composition on the right by the operator R_r gives us

$$R_lTR_r = (I - A_l)R_r = R_r - A_lR_r,$$

and

$$TR_r = I - A_r.$$

The composition on the left by the operator R_l gives us

$$R_lTR_r = R_l(I - A_l) = R_l - R_lA_l.$$

Whence, the difference between the operators

$$R_lTR_r = R_r - A_lR_r,$$

and

$$R_lTR_r = R_l - R_lA_r,$$

implies that, the operator

$$R_l - R_r = A_lR_r - R_lA_r,$$

is a compact operator as a sum of two compact operators.

Left equivalent regularizator

Let T be a bounded linear operator defined from a normed space E into a normed space F. Saying a left equivalent regularizator of T the left regularizator R_l such that, the original equation

$$T\varphi = f,$$

and the left regularizing equation

$$R_lT\varphi = \varphi - A_l\varphi = R_lf,$$

admit the same solutions.

Theorem 2

The left regularizator R_l *is a left equivalent regularizator, if and only if* R_l *is injective.*

Proof

- *First case*

Let φ be a solution of the left regularizing equation and the original equation. Say

$$R_l T\varphi = R_l f \ ,$$

and

$$T\varphi = f.$$

Whence, we write******************

$$\begin{aligned} R_l f = 0 &\Leftrightarrow R_l T\varphi = 0 \\ T\varphi = 0 &\Leftrightarrow f = 0 \\ &\Leftrightarrow N(R_l) = \{0\}. \end{aligned}$$

Hence, it follows that, R_l is injective.

- *Second case*

Let φ be a solution of the left regularizing equation. Say

$$R_l T\varphi = R_l f \ \Leftrightarrow R_l\left(T\varphi - f\right) = 0.$$

The operator R_l is injective, then

$$\begin{aligned} N(R_l) = \{0\} &\Rightarrow R_l(T\varphi - f) = 0 \\ &\Leftrightarrow \\ T\varphi - f \ = 0 &\Rightarrow T\varphi = f \ . \end{aligned}$$

Hence, φ is a solution of the original equation.

Right equivalent regularizator

Let T be a bounded linear operator defined from a normed space E into a normed space F. Saying a right equivalent regularizator of T the right regularizator R_r such that, the solution φ of the original equation

$$T\varphi = f,$$

is a range by the right regularizator R_r of the solution ψ of the right regularizing equation

$$TR_r\psi = \varphi - A_l\varphi = R_l f.$$

In other words, we get

$$\varphi = R_r\psi.$$

Theorem 3

The right regularizator R_r is a right equivalent regularizator, if and only if R_r is surjective.

Proof

- *First case*

Let φ be a solution of the original equation. Say

$$T\varphi = f.$$

The solution φ is a range by the right regularizator R_r of the solution ψ of the right regularizing equation

$$TR_r\psi = \psi - A_r\psi = f,$$

this implies that, for all solution φ of the original equation there exists a solution ψ of the right regularizing equation such that

$$\varphi = R_r\psi.$$

Hence, it follows that, R_r is surjective.

- *Second case*

Let φ be a solution of the original equation. Say

$$T\varphi = f.$$

The operator R_r is surjective, then there exists an element ψ such that

$$\varphi = R_r\psi,$$

or still

$$TR_r\psi = f$$

Hence, ψ is a solution of the right regularizing equation.

Bibliography

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